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ORIGINAL PAPER

# Permanence and partial extinction in an impulsive delay competitive system with the effect of toxic substances

**Zhijun Liu · Jing Hui · Jianhua Wu**

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**Abstract** In this paper we propose a periodic impulsive delay two-species competitive system in which two species have toxic inhibitory effects on each other. It is assumed that the system is impulsively controlled by means of harvesting and stocking controls. By using the theory of impulsive differential equation and analysis techniques, a set of sufficient conditions are derived for the permanence and partial extinction of the system. It turns out that the impulsive controls play a crucial role in shaping the above dynamics of the system. Numerical simulations are presented to substantiate the analytical results.

**Keywords** Permanence and partial extinction · Competitive system · Impulse · Delay · Toxicant

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Z. Liu (✉) · J. Wu (✉)
College of Mathematics and Information Science, Shaanxi Normal University Xi'an,
Shaanxi 710062, People's Republic of China
e-mail: zhijun_liu47@hotmail.com

J. Wu
e-mail: jianhuaw@snnu.edu.cn

Z. Liu
Department of Mathematics, Hubei Institute for Nationalities Enshi, Hubei 445000,
People's Republic of China

J. Hui
Department of Information and Computation Sciences, Guangxi University of Technology,
Liuzhou 545006, People's Republic of China
e-mail: gxuthui@163.com

## 1 Introduction

With the rapid development of modern industry and agriculture, a great quantity of toxic substances enter into ecosystems one after another. These pollutants seriously threaten the survival of the exposed populations. Therefore, it is important to study various ecosystems with the effect of toxins and to find a theoretical threshold value, which determines permanence or extinction of a population or community.

In recent years many investigations have been conducted to study the effect of toxic substances on biological species by using mathematical models (see [1–14]). In particular, Maynard Smith [3] incorporated the effect of toxic substances in a two-species Lotka–Volterra competitive system by assuming that each species produces a substance toxic to the other, but only when the other is present. Mukhopadhyay et al. [5] suggested that a species needs some time to mature before producing a substance which is toxic to the other, i.e., the production of a toxic substance by the competing species is not instantaneous, but occurs after some discrete time lag required for maturity of the species. Furthermore, considering the periodic environmental factor, Song and Chen [13] investigated the following periodic two-species competitive system with the effect of toxic substances

$$\begin{aligned} x'(t) &= x(t)[K_1(t) - \alpha_1(t)x(t) - \beta_1(t)y(t) - \gamma_1(t)x(t)y(t-\tau_1(t))], \\ y'(t) &= y(t)[K_2(t) - \alpha_2(t)y(t) - \beta_2(t)x(t) - \gamma_2(t)x(t-\tau_2(t))y(t)]. \end{aligned} \tag{1.1}$$

Here $x(t)$, $y(t)$ stand for the population densities of two competing species; $K_1(t)$, $K_2(t)$ are the intrinsic growth rates of two competing species; $\alpha_i(t)(i = 1, 2)$ are the coefficients of intraspecific competition; $\beta_i(t)(i = 1, 2)$ are the coefficients of interspecific competition; $K_i(t)/\alpha_i(t)(i = 1, 2)$ are the environmental carrying capacities of two competing species; $\gamma_1(t)$ and $\gamma_2(t)$ are, respectively, the rates of toxic inhibition of the species $x$ by the species $y$ and vice versa.

In addition, the ecological systems are often deeply perturbed by human activities such as harvesting and stocking. Usually, these activities are considered continuously by adding some items in these systems [15,16]. However, it is not always realistic. Mostly, the harvesting or stocking of the species is seasonal or occurs in regular pulses. These activities may also be periodic naturally, for example, a fisherman may go fishing at the same time once a day or once a week. It is evident that the species population levels repeatedly undergo changes of relatively short duration at fixed times due to harvesting or stocking. Because the duration of these changes is often negligible compared to the total duration of the life process, such changes can be well-approximated as being instantaneous changes of state, or impulses. These systems tend to be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. Impulsive differential equations are found in almost every domain of applied sciences, and numerous examples are given in Bainov and his collaborators' books [17,18]. Some impulsive equations have been recently introduced in population dynamics, such as vaccination [19], population ecology [20–22], chemotherapeutic treatment of disease [23], the chemostat [24–26], birth pulses [27–29].

Considering the possibility of impulsive perturbations, we introduce impulses in system (1.1) and obtain the following periodic impulsive delay competitive system with the effect of toxic substances

$$\begin{cases} \left.\begin{aligned} x'(t) &= x(t)[K_1(t)-\alpha_1(t)x(t)-\beta_1(t)y(t)-\gamma_1(t)x(t)y(t-\tau_1(t))], \\ y'(t) &= y(t)[K_2(t)-\alpha_2(t)y(t)-\beta_2(t)x(t)-\gamma_2(t)y(t)x(t-\tau_2(t))], \end{aligned}\right\}, \quad t \neq t_k, \\ \left.\begin{aligned} x(t_k^+) &= x(t_k)+p, \\ y(t_k^+) &= (1+b_k)y(t_k), \end{aligned}\right\}, \quad t = t_k, \quad k \in \mathbf{N} \end{cases} \tag{1.2}$$

with initial condition

$$\begin{aligned} (x(s), y(s)) = \phi(s) = (\phi_1(s), \phi_2(s)), \ \ \text{for} \ -\tau \le s \le 0, \\ \phi(0) > 0, \phi \in PC([-\tau, 0], \mathbf{R}_+^2). \end{aligned} \tag{1.3}$$

Here $\tau = \max_{1\le i\le 2}\max_{t\in[0,\omega]}\{\tau_i(t)\}.K_i(t), \alpha_i(t), \beta_i(t), \gamma_i(t), \tau_i(t)$ are continuous $\omega$-periodic functions, and $\alpha_i(t), \beta_i(t), \gamma_i(t)$ are positive and $\tau_i(t)(i = 1, 2)$ are nonnegative. The intrinsic growth rates $K_i(t)(i = 1, 2)$ are not necessarily positive, since the environment fluctuates randomly, in weak environment, $K_i(t)(i = 1, 2)$ may be negative. $\mathbf{N}$ is the set of positive integers. The jump conditions reflect the possibility of impulsive effects on the species $x$ and $y$. $p > 0$ is the impulsive stocking amount of the species $x$ at $t = t_k$, which implies that the populations are subjected to impulsive stocking at a constant rate $p$. $b_k y(t_k) < 0$ may represent the impulsive harvesting amount of the species $y$ at $t = t_k$, while $b_k y(t_k) > 0$, the perturbations may stand for the impulsive stocking amount of the species $y$ at $t = t_k$. That is, the impulsive harvesting amount (or stocking amount) is proportional to the current density of the species $y$. As we know, maintaining ecological balance is an important ecological problem for a population or community. It is quite reasonable to consider the problem: if we wish to make the population coexist or extinct, how should we control the harvesting or stocking effort?

Despite the apparent abundance of applications, the study of periodic impulsive delay equations is in its relative infancy. To our knowledge, few papers have been published on the permanence and partial extinction of system (1.2). The main purpose of this paper is to derived a set of easily verifiable sufficient conditions for the permanence and partial extinction of system (1.2). This paper is organized as follows. In Sect. 2, we give some assumptions, notations and preliminary lemmas which are used in the later sections. System (1.2) is analyzed to find sufficient conditions for the permanence and partial extinction of two species in Sect. 3. In the final section, we conclude our paper and present specific examples and their numerical simulations to interpret how impulsive harvesting or stocking policy can lead to significant changes in size of two species.

## 2 Preliminaries

In this section, we shall introduce some assumptions, notations and state preliminary lemmas which will be useful for establishing our main results.

In system (1.2), we will use the following assumptions:

- (H1) $0 \doteq t_0 < t_1 < t_2 < \cdots$ are fixed impulsive points with $\lim_{k\to\infty} t_k = \infty$;
- (H2) $\{b_k\}$ is a real sequence and $b_k > -1, k \in \mathbf{N}$;
- (H3) $p > 0$ and there exists an integer $q > 0$ such that $b_{k+q} = b_k, t_{k+q} = t_k + \omega$;
- (H4) $\tau_i(t) \in C([0, \infty), [0, \infty))$ are $\omega$-periodic functions and $t - \tau_i(t) \to \infty$ as $t \to \infty, i = 1, 2$.

In the following, we introduce some notations and lemmas.
Let $\mathbf{J} \subset \mathbf{R}$ and $\mathbf{J} = [t_0, \infty)$. We introduce the following spaces of functions:
$PC(\mathbf{J}, \mathbf{R}) \doteq \{u : \mathbf{J} \to \mathbf{R}$ : is continuous for $t \in \mathbf{J}, t \neq t_k$, and has discontinuities of the first kind at the points $t_k \in \mathbf{J}$ $(k \in \mathbf{N})$ where they are continuous from the left$\}$ and
$PC^1(\mathbf{J}, \mathbf{R}) \doteq \{u \in PC(\mathbf{J}, \mathbf{R}) : u$ is continuously differential for $t \in \mathbf{J}, t \neq t_k, u'(t_k^+)$ and $u'(t_k^-)$ exist, $k \in \mathbf{N}\}$.

**Lemma 2.1** (Impulsive Differential Inequalities, Theorem 1.4.1 in [18]) *If the function $u \in PC^1([t_0, \infty), \mathbf{R})$ satisfies the inequalities*

$$\begin{cases} u'(t) \le p(t)u(t) + f(t), & t \neq t_k, \quad t \ge t_0, \\ u(t_k^+) \le d_k u(t_k) + h_k, & k \in \mathbf{N}, \end{cases} \tag{2.1}$$

*where $p(t), f(t) \in PC([t_0, \infty), \mathbf{R}), d_k > 0, h_k$ are constants for $k \in \mathbf{N}$. Then for $t \ge t_0$*

$$u(t) \le u(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) h_k$$
$$+ \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(r)dr\right) f(s)ds.$$

*Analogously, we have*

$$u(t) \ge u(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) h_k$$
$$+ \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(r)dr\right) f(s)ds.$$

*for $t \ge t_0$ if all the inequalities of (2.1) are inverse.*

**Lemma 2.2** (Lemma 2.4 in [30]) *Consider the following impulsive system*

$$\begin{cases} x'(t) = \alpha(t)x(t), & t \neq t_k, \\ x(t_k^+) = x(t_k) + p, \end{cases} \tag{2.2}$$

*where $\alpha(t)$ is a continuous $\omega$-periodic function, $p$ is a positive constant and there is an integer $q > 0$ such that $t_{k+q} = t_k + \omega$. If $\int_0^{\omega} \alpha(t)dt < 0$, then (2.2) has a unique positive, globally asymptotically stable $\omega$-periodic solution.*

**Lemma 2.3** (Lemma 2.3 in [21]) *Let $u(t)$ be a continuous $\omega$-periodic function. If $\omega > 0$ and $\int_0^{\omega} u(t)dt > 0$, then the following inequality*

$$\exp\{c(t-s)\} \leq \exp\left\{1 + d\omega + \int_s^t u(r)dr\right\}, \quad \textit{for } t \geq s, t \to \infty \tag{2.3}$$

*holds, where $d = \max_{t\in[0,\omega]}\{|u(t)|\}$ and the constant $c$ satisfies $0 < c \leq \min\{\int_0^{\omega} u(t)dt/\omega, 1/\omega\}$.*

*Similar to Lemma 2.1 in [31] or Theorem 2.1 in [20], we can obtain Lemma 2.4.*

**Lemma 2.4** *Consider the following impulsive system*

$$\begin{cases} x'(t) = x(t)(\alpha(t) - \beta(t)x(t)), & t \neq t_k, \\ x(t_k^+) = x(t_k) + p. \end{cases} \tag{2.4}$$

*where $\alpha(t)$, $\beta(t)$ are continuous $\omega$-periodic functions and $\beta(t) > 0$, $p$ is a positive constant and there is an integer $q > 0$ such that $t_{k+q} = t_k + \omega$. If $\int_0^{\omega} \alpha(t)dt > 0$, then system (2.4) has a unique positive, globally asymptotically stable $\omega$-periodic solution.*

**Lemma 2.5** (Lemma 2.2 in [30]) *Consider the following impulsive system*

$$\begin{cases} y'(t) = y(t)(\alpha(t) - \beta(t)y(t)), & t \neq t_k, \\ y(t_k^+) = (1 + b_k)y(t_k). \end{cases} \tag{2.5}$$

*where $\alpha(t)$, $\beta(t)$ are continuous $\omega$-periodic functions and $\beta(t) > 0$, and there exists an integer $q > 0$ such that $b_{k+q} = b_k$, $t_{k+q} = t_k + \omega$ and $b_k > -1$ for $k \in \mathbf{N}$. Then*

*(1) any solution $y(t)$ of (2.5) with positive initial value satisfies $\lim_{t\to\infty} y(t) = 0$ if*

$$\prod_{k=1}^{q}(1 + b_k)\exp\left\{\int\limits_0^{\omega} \alpha(t)\right\} < 1; \tag{2.6}$$

*(2) (2.5) has a unique positive $\omega$-periodic solution if and only if*

$$\prod_{k=1}^{q}(1 + b_k)\exp\left\{\int\limits_0^{\omega} \alpha(t)\right\} > 1. \tag{2.7}$$

*Moreover, the unique positive $\omega$-periodic solution is globally asymptotically stable.*

## 3 Permanence and partial extinction

Denote

$$\begin{aligned}\lambda_1 &\doteq \prod_{k=1}^{q}(1+b_k)\exp\left\{\int_0^{\omega}[K_2(t)-\beta_2(t)\bar{v}_1(t)]dt\right\},\\ \lambda_2 &\doteq \prod_{k=1}^{q}(1+b_k)\exp\left\{\int_0^{\omega}[K_2(t)-\beta_2(t)\bar{v}_2(t)]dt\right\},\end{aligned} \tag{3.1}$$

where $\bar{v}_1(t)$ and $\bar{v}_2(t)$ are, respectively, the unique positive globally asymptotically stable $\omega$-periodic solutions of systems

$$\begin{cases} v'(t) = K_1(t)v(t), \quad t \neq t_k,\\ v(t_k^+) = v(t_k) + p,\\ \text{when} \int_0^{\omega} K_1(t)dt < 0 \end{cases} \tag{3.2}$$

and

$$\begin{cases} v'(t) = v(t)[K_1(t) - \alpha_1(t)v(t)], \quad t \neq t_k,\\ v(t_k^+) = v(t_k) + p,\\ \text{when} \int_0^{\omega} K_1(t)dt > 0. \end{cases} \tag{3.3}$$

If $\lambda_1 > 1$ and $\lambda_2 > 1$, then we can choose a constant $\varepsilon$ such that

$$0 < \varepsilon < \min\left\{\frac{\ln \lambda_1}{\int_0^{\omega}\beta_2(t)dt}, \frac{\ln \lambda_2}{\int_0^{\omega}\beta_2(t)dt}\right\}. \tag{3.4}$$

Now, we state the theorem for the permanence of two species.

**Theorem 3.1** *For system* (1.2) *we have the following conclusions:*

(1) *the species x is permanent;*
(2) *when $\int_0^{\omega} K_1(t)dt < 0$, if $\lambda_1 > 1$, then the species y is permanent;*
(3) *when $\int_0^{\omega} K_1(t)dt > 0$, if $\lambda_2 > 1$, then the species y is permanent.*

*Proof* (1) We first show that $y(t)$ is uniformly ultimately upper bounded. From (1.2), we have

$$\begin{cases} \left.\begin{array}{l} x(t) = x(t_{k-1}^{+}) \exp\left\{ \int\limits_0^t [K_1(s) - \alpha_1(s)x(s) - \beta_1(s)y(s) \right. \\ \qquad \left. -\gamma_1(s)x(s)y(s-\tau_1(s))]ds \right\}, \\ y(t) = y(t_{k-1}^{+}) \exp\left\{ \int\limits_0^t [K_2(s) - \alpha_2(s)y(s) - \beta_2(s)x(s) \right. \\ \qquad \left. -\gamma_2(s)y(s)x(s-\tau_2(s))]ds \right\}, \end{array}\right\}, \quad t \in (t_{k-1}, t_k]. \\ \left.\begin{array}{l} x(t_k^{+}) = x(t_k) + p, \\ y(t_k^{+}) = (1+b_k)y(t_k), \end{array}\right\}, \quad t = t_k. \end{cases}$$

It is easy to prove that $x(t) > 0$, $y(t) > 0$ if $x(0) > 0$, $y(0) > 0$. So we get

$$\begin{cases} y'(t) \le K_2(t)y(t), \quad t \ne t_k, \\ y(t_k^{+}) = (1+b_k)y(t_k) \end{cases} \tag{3.5}$$

and

$$\begin{cases} y'(t) \le y(t)[K_2(t) - \alpha_2(t)y(t)], \quad t \ne t_k, \\ y(t_k^{+}) = (1+b_k)y(t_k). \end{cases} \tag{3.6}$$

To obtain the uniformly ultimately upper bound of $y(t)$, we consider **Cases** $(A_1)$ and $(A_2)$.

**Case** $(A_1)$ $\prod_{k=1}^{q}(1+b_k)\exp\{\int_0^\omega K_2(t)dt\} \le 1$.
From Lemma 2.1 and (3.5), it follows that

$$y(t) \le y(0) \prod_{0<t_k<t} (1+b_k) \exp\left( \int_0^t K_2(s)ds \right). \tag{3.7}$$

When $t \in (n\omega, (n+1)\omega]$, $n \in \mathbf{N} \cup \{0\}$, let

$$\mathscr{B} = \max_{s\in[0,\omega]} \prod_{0\le t_k<s} (1+b_k), \quad \mathscr{K} = \max_{t\in[n\omega,(n+1)\omega]} \exp\left( \int_{n\omega}^t K_2(s)ds \right).$$

From (3.7), we get

$$\begin{aligned} y(t) &\le y(0) \prod_{0<t_k<n\omega} (1+b_k) \prod_{n\omega\le t_k<t} (1+b_k) \exp\left( \int_0^t K_2(s)ds \right) \\ &= y(0) \left( \prod_{0<t_k<\omega} (1+b_k) \exp\left( \int_0^\omega K_2(s)ds \right) \right)^n \\ &\quad \prod_{0<t_k<t-n\omega} (1+b_k) \exp\left( \int_{n\omega}^t K_2(s)ds \right) \\ &= y(0) \left( \prod_{k=1}^q (1+b_k) \exp\left( \int_0^\omega K_2(s)ds \right) \right)^n \mathscr{B}\mathscr{K}. \end{aligned}$$

If $\prod_{k=1}^q (1+b_k) \exp\left(\int_0^\omega K_2(s)ds\right) = 1$, then we have $y(t) \le y(0)\mathscr{B}\mathscr{K}$. If $\prod_{k=1}^q (1+b_k) \exp(\int_0^\omega K_2(s)ds) < 1$, then we have

$$\lim_{n\to\infty} y(0) \left( \prod_{i=1}^q (1+b_k) \exp\left( \int_0^\omega K_2(s)ds \right) \right)^n \mathscr{B}\mathscr{K} = 0. \tag{3.8}$$

Hence $\lim_{t\to\infty} y(t) = 0$, which implies that there exist $a_1 > 0$ and $T_1 > 0$ such that

$$y(t) \le a_1, \quad \text{for } t \ge T_1. \tag{3.9}$$

**Case** $(A_2)$ $\prod_{k=1}^q (1+b_k) \exp\{\int_0^\omega K_2(t)\} > 1$.
Consider the comparison system of (3.6)

$$\begin{cases} u'(t) = u(t)[K_2(t) - \alpha_2(t)u(t)], & t \ne t_k, \\ u(t_k^+) = (1+b_k)u(t_k). \end{cases} \tag{3.10}$$

From (2) in Lemma 2.5, it follows that (3.10) has a unique positive, globally asymptotically stable $\omega$-periodic solution denoted by $\bar{u}_1(t)$. The global attractivity of $\bar{u}_1(t)$ implies that there exist $T_2 > 0$ and $a_2 > 0$ such that

$$u(t) \le \bar{u}_1(t) + a_2, \quad \text{for } t \ge T_2. \tag{3.11}$$

Let $u(t)$ be any solution of (3.10) with $u(0) = y(0) > 0$. From (3.6) and (3.10), we use the comparison theorem of impulsive differential equation, and then obtain that $y(t) \le u(t)$ for $t \ge 0$. From this and (3.11) one has

$$y(t) \le \bar{u}_1(t) + a_2, \quad \text{for } t \ge T_2. \tag{3.12}$$

It follows from *Cases* $(A_1)$ and $(A_2)$ that there exist $T_3 = \max\{T_1, T_2\} > 0$ and $M_y = \max_{t\in[0,\omega]}\{y(0)\mathscr{B}\mathscr{K}, a_1, \bar{u}_1(t) + a_2\} > 0$ such that

$$y(t) \le M_y, \quad \text{for } t \ge T_3. \tag{3.13}$$

Next, we show that $x(t)$ is uniformly ultimately upper bounded. Similar to (3.5) and (3.6), from (1.2) we have

$$\begin{cases} x'(t) \le K_1(t)x(t), & t \ne t_k, \\ x(t_k^+) = x(t_k) + p, \end{cases} \tag{3.14}$$

and

$$\begin{cases} x'(t) \le x(t)[K_1(t) - \alpha_1(t)x(t)], & t \ne t_k, \\ x(t_k^+) = x(t_k) + p. \end{cases} \tag{3.15}$$

To obtain the uniformly ultimately upper bound of $x(t)$, we consider **Cases** $(B_1)$ and $(B_2)$.

**Case** $(B_1)$ $\int_0^\omega K_1(t)dt < 0$.
From Lemma 2.2, it follows that the comparison system of (3.14)

$$\begin{cases} v'(t) = K_1(t)v(t), & t \ne t_k, \\ v(t_k^+) = v(t_k) + p \end{cases} \tag{3.16}$$

has a unique positive, globally asymptotically stable $\omega$-periodic solution denoted by $\bar{v}_1(t)$. The global attractivity of $\bar{v}_1(t)$ implies that there exist $\varepsilon > 0$ and $T_4 > 0$ such that

$$v(t) \le \bar{v}_1(t) + \varepsilon, \quad \text{for } t \ge T_4,$$

where the constant $\varepsilon$ satisfies (3.4). From this, (3.14) and (3.16), we have

$$x(t) \le \bar{v}_1(t) + \varepsilon, \quad \text{for } t \ge T_4. \tag{3.17}$$

**Case** $(B_2)$ $\int_0^\omega K_1(t)dt \ge 0$.
We can choose a constant $\delta > 0$ such that $\int_0^\omega (K_1(t) + \delta)dt > 0$. By Lemma 2.4, we obtain that the comparison system of (3.15)

$$\begin{cases} v'(t) = v(t)[K_1(t) + \delta - \alpha_1(t)v(t)], & t \ne t_k, \\ v(t_k^+) = v(t_k) + p \end{cases} \tag{3.18}$$

has a unique positive, globally asymptotically stable $\omega$-periodic solution denoted by $\bar{v}_2(t)$. Similar to *Case* $(B_1)$, we can easily obtain that there exist $\varepsilon > 0$ and $T_5 > 0$

such that

$$x(t) \le \bar{v}_2(t) + \varepsilon, \quad \text{for } t \ge T_5, \tag{3.19}$$

where the constant $\varepsilon$ satisfies (3.4). Let $M_x = \max_{t\in[0,\omega]}\{\bar{v}_1(t) + \varepsilon, \bar{v}_2(t) + \varepsilon\}$ and $T_6 = \max\{T_4, T_5\}$. Then from *Cases* $(B_1)$ and $(B_2)$ we have

$$x(t) \le M_x, \quad \text{for } t \ge T_6. \tag{3.20}$$

Finally, we prove that $x(t)$ is uniformly ultimately lower bounded. From (3.13) and (3.20) we have

$$x(t) \le M_x, \quad y(t) \le M_y, \quad \text{for } t > T_7 = \max\{T_3, T_6\}. \tag{3.21}$$

So from (1.2) and (3.21) there exists $T_8 \doteq T_7 + \tau$ such that for $t \ge T_8$

$$\begin{cases} x'(t) & \ge x(t)(K_1(t) - \alpha_1(t)M_x - \beta_1(t)M_y - \gamma_1(t)M_xM_y) \\ & \ge x(t)(K_1(t) - M_1 - \alpha_1(t)M_x - \beta_1(t)M_y - \gamma_1(t)M_xM_y), \quad t \ne t_k, \\ x(t_k^+) = x(t_k) + p, \end{cases} \tag{3.22}$$

where the constant $M_1 > 0$ satisfies

$$\int_0^\omega (K_1(t) - M_1 - \alpha_1(t)M_x - \beta_1(t)M_y - \gamma_1(t)M_xM_y)dt < 0.$$

It follows from Lemma 2.2 that the comparison system

$$\begin{cases} v'(t) = v(t)(K_1(t) - M_1 - \alpha_1(t)M_x - \beta_1(t)M_y - \gamma_1(t)M_xM_y), \quad t \ne t_k, \\ v(t_k^+) = v(t_k) + p \end{cases} \tag{3.23}$$

has a unique positive, globally asymptotically stable $\omega$-periodic solution denoted by $\bar{v}_3(t)$. The global attractivity of $\bar{v}_3(t)$ implies that there exist $\epsilon_1 > 0$ and $T_9 > T_8$ such that

$$v(t) \ge \bar{v}_3(t) - \epsilon_1, \quad \text{for } t \ge T_9, \tag{3.24}$$

where $\epsilon_1 < \min_{t\in[0,\omega]} \frac{1}{2}\bar{v}_3(t)$. We denote the solution of (3.23) satisfying $v(T_8) = x(T_8)$ by $v(t)$. By the comparison theorem of impulsive differential equation and (3.22)–(3.24), we have

$$x(t) \ge \bar{v}_3(t) - \epsilon_1 \ge \min_{t\in[0,\omega]}\{\bar{v}_3(t)\} - \epsilon_1 \doteq m_x, \quad \text{for } t \ge T_9. \tag{3.25}$$

It follows from (3.20) and (3.25) that the species $x$ is permanent.

(2) From (3.13), it is easy to see that $y(t)$ is uniformly ultimately upper bounded. Next, we show that $y(t)$ is uniformly ultimately lower bounded.

When $\int_0^\omega K_1(t)dt < 0$, for $t \geq T_4 + \tau$, it follows from (1.2) and (3.17) that

$$\begin{cases} y'(t) \geq y(t)[K_2(t) - \beta_2(t)(\bar{v}_1(t)+\varepsilon) - (\alpha_2(t)+\gamma_2(t)(\bar{v}_1(t)+\varepsilon))y(t)], & t \neq t_k, \\ y(t_k^+) = (1+b_k)y(t_k). \end{cases} \tag{3.26}$$

Consider the comparison system of (3.26)

$$\begin{cases} u'(t) = u(t)[K_2(t) - \beta_2(t)(\bar{v}_1(t)+\varepsilon) - (\alpha_2(t)+\gamma_2(t)(\bar{v}_1+\varepsilon))y(t)], & t \neq t_k, \\ u(t_k^+) = (1+b_k)u(t_k). \end{cases} \tag{3.27}$$

Note that the choice of $\varepsilon$ (see 3.4), it is easy to verify that $\prod_{k=1}^q (1+b_k)\exp\{\int_0^\omega (K_2(t) - \beta_2(t)(\bar{v}_1(t) + \varepsilon))dt\} > 1$, then it follows from Lemma 2.5 that (3.27) has a unique positive, globally asymptotically stable $\omega$-periodic solution denoted by $\bar{u}_2(t)$. Similar to (3.24)–(3.25), we obtain that there exist $\epsilon_2 > 0$ and $T_{10} \geq T_4 + \tau$ such that

$$y(t) \geq \bar{u}_2(t) - \epsilon_2 \geq \min_{t \in [0,\omega]} \{\bar{u}_2(t)\} - \epsilon_2 \doteq m_y, \quad \text{for } t \geq T_{10}, \tag{3.28}$$

where $\epsilon_2 < \min_{[0,\omega]} \frac{1}{2}\bar{u}_2(t)$. It follows from (3.13) and (3.28) that the species $y$ is permanent.

(3) When $\int_0^\omega K_1(t)dt \geq 0$, the proof is similar to that of the case (2), and the details are omitted. The proof of Theorem 3.1 is complete. □

In the following, we present the theorem for the permanence of the species $x$ and the extinction of the species $y$.

**Theorem 3.2** *If*

$$\prod_{k=1}^{q} (1 + b_k) \exp\left(\int_0^\omega K_2(s)ds\right) \leq 1, \tag{3.29}$$

*then any positive solution* $(x(t), y(t))$ *of (1.2) satisfies*

$$\lim_{t\to\infty} |x(t) - \bar{\chi}(t)| = 0, \quad \lim_{t\to\infty} y(t) = 0,$$

*where* $\bar{\chi}(t)$ *is a unique positive* $\omega$*-periodic solution of the impulsive logistic system*

$$\begin{cases} \chi'(t) = \chi(t)(K_1(t) - \alpha_1(t)\chi(t)), & t \neq t_k, \\ \chi(t_k^+) = \chi(t_k) + p. \end{cases} \tag{3.30}$$

*Proof* We first consider the global attractivity of $y(t)$. From (1) in Theorem 3.1 we know that $x$ is permanent. In particular, from (3.20) and (3.25) we have

$$m_x \le x(t) \le M_x, \quad t \ge T_9. \tag{3.31}$$

From (1.2), there exists $t_* \doteq T_9 + \tau$ such that for $t \ge t_*$

$$y'(t) \le y(t)[K_2(t) - \beta_2(t)m_x - (\alpha_2(t) + \gamma_2(t)m_x)y(t)]. \tag{3.32}$$

From (3.29) we obtain that $\prod_{k=1}^{q}(1+b_k)\exp(\int_0^\omega (K_2(s) - \beta_2(s)m_x)ds < 1$. So from Lemma 2.5 we have that any positive solution of the comparison system

$$\begin{cases} u'(t) = u(t)[K_2(t) - \beta_2(t)m_x - (\alpha_2(t) + \gamma_2(t)m_x)y(t)], \quad t \ne t_k, \\ u(t_k^+) = (1+b_k)u(t_k) \end{cases} \tag{3.33}$$

satisfies $\lim_{t\to\infty} u(t) = 0$. By the comparison theorem of impulsive differential equation, we obtain that $\lim_{t\to\infty} y(t) = 0$.

In the following, we investigate the global attractivity of $x(t)$. We consider two cases.

**Case (1)** $\int_0^\omega K_1(s)ds \le 0$.
Since $K_1(t)$ is a continuous $\omega$-periodic function with $\int_0^\omega K_1(s)ds \le 0$, there exists a constant $\delta > 0$ such that

$$\int_s^t K_1(r)dr \le \delta, \quad \text{for } t \ge s, t \to \infty. \tag{3.34}$$

Let $z(t) = x(t) - \bar{\chi}(t)$, then

$$\begin{cases} z'(t) = z(t)[K_1(t) - \alpha_1(t)(x(t) + \bar{\chi}(t)) \\ \quad -\beta_1(t)y(t) - \gamma_1(t)x(t)y(t - \tau_1(t))], \quad t \ne t_k, \\ z(t_k^+) = x(t_k) + p - (\bar{\chi}(t_k) + p) = x(t_k) - \bar{\chi}(t_k) = z(t_k). \end{cases} \tag{3.35}$$

For $t \ge t_*$, we have

$$\begin{aligned} z(t) = z(t_*)\exp\left(\int_{t_*}^t (K_1(s) - \alpha_1(s)(x(s) + \bar{\chi}(s)))ds\right) \\ - \int_{t_*}^t [\beta_1(s)y(s) + \gamma_1(s)x(s)y(t - \tau_1(s))] \\ \exp\left(\int_s^t (K_1(r) - \alpha_1(r)(x(r) + \bar{\chi}(r)))dr\right) ds. \end{aligned}$$

Furthermore, one obtains that for $t \to \infty$,

$$
\begin{aligned}
|z(t)| \le{}& |z(t_*)| \exp\left(\int_{t_*}^{t} K_1(s)ds\right) \exp\left(\int_{t_*}^{t} (-m_x\alpha_1(s))ds\right) \\
&+ \int_{t_*}^{t} [\beta_1^U y(s) + \gamma_1^U M_x y(s-\tau_1(s))] \exp\left(\int_{s}^{t} K_1(r)dr\right) \\
&\times \exp\left(\int_{s}^{t} (-m_x\alpha_1(r))dr\right) ds \\
\le{}& |z(t_*)| \exp\{1+\delta+d_1\omega - c_1(t-t_*)\} \\
&+ \exp\{1+\delta+d_1\omega\} \int_{t_*}^{t} [\beta_1^U y(s) + \gamma_1^U M_x y(s-\tau_1(s))] \exp\{c_1(s-t)\}ds,
\end{aligned}
\tag{3.36}
$$

where $\beta_1^U = \max_{t\in[0,\omega]} \beta_1(t)$, $\gamma_1^U = \max_{t\in[0,\omega]} \gamma_1(t)$, $0 < c_1 < \min\{\int_0^\omega m_x\alpha_1(t)\, dt/\omega, 1/\omega\}$ and $d_1 = \max_{t\in[0,\omega]}\{m_x\alpha_1(t)\}$. Since $\lim_{t\to\infty} y(t) = 0$, it follows from the assumption (H4) that we have $\lim_{t\to\infty} y(t-\tau_1(t)) = 0$. A simple calculation shows that

$$
\begin{aligned}
&\lim_{t\to\infty} \int_{t_*}^{t} y(s)\exp\{c_1(s-t)\}ds = 0, \\
&\lim_{t\to\infty} \int_{t_*}^{t} y(s-\tau_1(s))\exp\{c_1(s-t)\}ds = 0.
\end{aligned}
\tag{3.37}
$$

Thus we obtain that $\lim_{t\to\infty} \int_{t_*}^{t} [\beta_1^U y(s) + \gamma_1^U M_x y(s-\tau_1(s))] \exp\{c_1(s-t)\}ds = 0$. It follows from (3.36) to (3.38) that $\lim_{t\to\infty} |z(t)| = 0$, that is,

$$
\lim_{t\to\infty} |x(t) - \bar{\chi}(t)| = 0. \tag{3.38}
$$

**Case (2)** $\int_0^\omega K_1(s)ds > 0$.

Making the change of variable $z(t) = x^{-1}(t) - \bar{\chi}^{-1}(t)$, one has

$$
\begin{cases}
z'(t) = -K_1(t)z(t) + \frac{\beta_1(t)y(t)}{x(t)} + \gamma_1(t)y(t-\tau_1(t)), \quad t \ne t_k, \\
z(t_k^+) = (x(t_k)+p)^{-1} - (\bar{\chi}(t_k)+p)^{-1} = (1+p\varrho_k)^{-2} z(t_k) \doteq \sigma_k z(t_k),
\end{cases}
\tag{3.39}
$$

where $\varrho_k$ is between $x^{-1}(t_k)$ and $\bar{\chi}^{-1}(t_k)$ and $\sigma_k = (1+p\varrho_k)^{-2}$.

For the time point $t_*$, we need to consider two cases: (1) $t_*$ is not an impulsive point; (2) $t_*$ is an impulsive point.

When $t_*$ is not an impulsive point, from (3.39) we have

$$\begin{aligned} z(t) &= z(t_*) \prod_{t_*<t_k<t} \sigma_k \exp\left(-\int_{t_*}^{t} K_1(s)ds\right) + \int_{t_*}^{t} \left\{ \prod_{s<t_k<t} \sigma_k \exp\left(-\int_{s}^{t} K_1(r)dr\right) \right. \\ &\quad \left. \times \left[\frac{\beta_1(s)y(s)}{x(s)} + \gamma_1(s)y(s-\tau_1(s))\right]\right\} ds. \end{aligned} \tag{3.40}$$

Using $0 < \sigma_k < 1$ and Lemma 2.3, we obtain that for $t \to \infty$,

$$\begin{aligned} |z(t)| &\le |z(t_*)| \exp\left(-\int_{t_*}^{t} K_1(s)ds\right) + \int_{t_*}^{t} \exp\left(-\int_s^t K_1(r)dr\right) \left[\frac{\beta_1(s)y(s)}{x(s)}\right. \\ &\quad \left. +\gamma_1(s)y(s-\tau_1(s))\right] ds \\ &\le |z(t_*)| \exp\{1 + d_2\omega - c_2(t-t_*)\} \\ &\quad + \exp\{1+d_2\omega\} \int_{t_*}^{t} \left[\frac{\beta_1^U y(s)}{m_x} + \gamma_1^U y(s-\tau_1(s))\right] \exp\{c_2(s-t)\} ds, \end{aligned} \tag{3.41}$$

where $0 < c_2 < \min\{\int_0^\omega K_1(t)dt/\omega, 1/\omega\}$ and $d_2 = \max_{t\in[0,\omega]}\{|K_1(t)|\}$. Similar to (3.36)–(3.38), we obtain that $\lim_{t\to\infty} z(t) = 0$. Using the mean value theorem, we obtain that $\lim_{t\to\infty} |x(t) - \bar{\chi}(t)| = 0$.

When $t_*$ is an impulsive point, from (3.39) we have

$$\begin{aligned} z(t) &= z(t_*^+) \left(\prod_{t_*^+<t_k<t} \sigma_k\right) \exp\left(-\int_{t_*^+}^{t} K_1(s)ds\right) \\ &\quad + \int_{t_*^+}^{t} \left\{ \prod_{s<t_k<t} \sigma_k \exp\left(-\int_{s}^{t} K_1(r)dr\right) \right. \\ &\quad \left. \times \left[\frac{\beta_1(s)y(s)}{x(s)} + \gamma_1(s)y(s-\tau_1(s))\right]\right\} ds. \end{aligned} \tag{3.42}$$

Similar to (3.40) and (3.41), one can easily show that $\lim_{t\to\infty} |x(t) - \bar{\chi}(t)| = 0$. This completes the proof. □

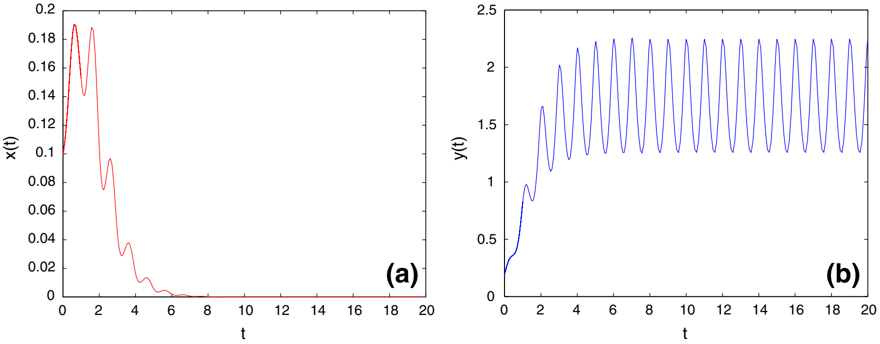

**Fig. 1** Asymptotic behaviors of system (4.1) with $K_1(t) = 1.1 + \sin 2\pi t$, $K_2(t) = 2 + \cos 2\pi t$, $p \equiv 0$, $b_k \equiv 0$. **a** Time-series of the species $x$. **b** Time-series of the species $y$

## 4 Numerical analysis and discussion

In this paper, we have investigated the dynamic behaviors of a periodic impulsive delay two-species competitive system with the effect of toxic substances. Each species is not isolated from its living environment, but competes with the other for the same resource. Moreover, it is affected by the human activities. So the model discussed in this paper is more realistic due to taking the discontinuity of human activities into account. Sufficient conditions for the permanence and partial extinction are established, respectively. The above results show that the impulsive controls play an important role while both permanence and partial extinction are independent of the values of delays and the toxic inhibition rates. These theoretical results are confirmed by the following examples and their numerical results.

As mentioned in the Introduction, system (1.2) can describe two competing species which are harvested or stocked seasonally. In fact, we can only take a finite number of harvests or stocks in a period (e.g., in one year). For simplicity, assume we harvest or stock once during a period. Now we consider the following impulsive delay systems and try to explain how the impulsive harvesting or stocking affects the two species' survivals. Consider

$$\begin{cases} \left.\begin{array}{l} x'(t) = x(t)[K_1(t) - (1.1 + \sin 2\pi t)x(t) - (1.15 + \sin 2\pi t)y(t) \\ \qquad\qquad -(1.2 + \cos 2\pi t)x(t)y(t-2)], \\ y'(t) = y(t)[K_2(t) - (1.2 + \cos 2\pi t)y(t) - (1.1 + \sin 2\pi t)x(t) \\ \qquad\qquad -(1.25 + \sin 2\pi t)y(t)x(t-1)], \end{array}\right\}, \quad t \neq t_k, \\ \left.\begin{array}{l} x(t_k^+) = x(t_k) + p, \\ y(t_k^+) = (1 + b_k)y(t_k), \end{array}\right\}, \quad t = t_k \end{cases} \tag{4.1}$$

with initial condition

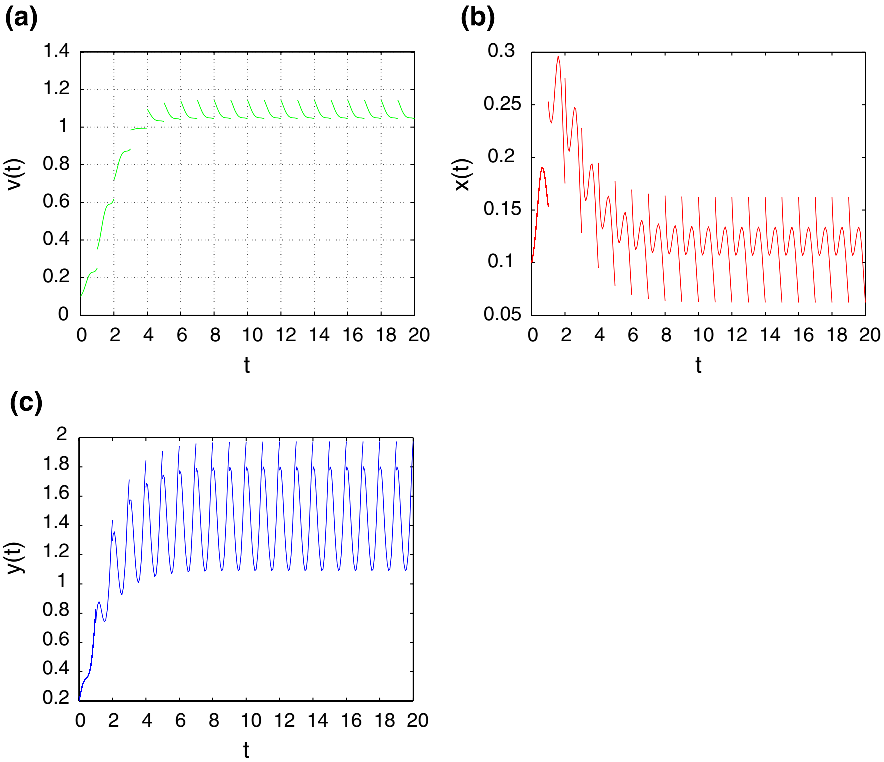

**Fig. 2** Asymptotic behaviors of systems (4.1) and (3.3) with $K_1(t) = 1.1 + \sin 2\pi t$, $K_2(t) = 2 + \cos 2\pi t$, $p \equiv 0.1$, $b_k \equiv -0.1$. **a** Time-series of the species $v$. **b** Time-series of the species $x$. **c** Time-series of the species $y$

$$\phi_1(t) = \begin{cases} 0, & t \in [-2, 0), \\ 0.1, & t = 0 \end{cases} \quad \text{and} \quad \phi_2(t) = \begin{cases} 0, & t \in [-2, 0), \\ 0.2, & t = 0. \end{cases} \tag{4.2}$$

We fix the coefficients that $\alpha_1(t) = 1.1 + \sin 2\pi t$, $\beta_1(t) = 1.15 + \sin 2\pi t$, $\gamma_1(t) = 1.2 + \cos 2\pi t$, $\alpha_2(t) = 1.2 + \cos 2\pi t$, $\beta_2(t) = 1.1 + \sin 2\pi t$, $\gamma_2(t) = 1.25 + \sin 2\pi t$, $\tau_1(t) = 2$, $\tau_2(t) = 1$, $\omega = 1$, $q = 1$, $t_k = k \in \mathbf{N}$. When $K_1(t) = 1.1 + \sin 2\pi t$, $K_2(t) = 2 + \cos 2\pi t$, the system (4.1) without impulsive effect ($p \equiv 0$, $b_k \equiv 0$) has an asymptotically stable semi-trivial solution (see Fig. 1, here the species $x$ is the inferior competitor). If we choose $p \equiv 0.1$, $b_k \equiv -0.1$, then system (3.3) has a unique positive globally asymptotically stable 1-periodic solution denoted by $\bar{v}_2(t)$ (see Fig. 2a, here $\bar{v}_2(t) < 1.2$), and the assumptions of (1) and (3) in Theorem 3.1 ($\lambda_2 > 1$) are satisfied, and from Fig. 2b and c we may observe that two species are permanent. So in this case the principle of competitive exclusion is invalid and the impulsive control strategy protects the inferior competitor $x$ from extinction. In fact, since the species $x$ is the inferior competitor without impulsive stocking, the competition ability of $x$ increases as $p$ increases if $b_k$ is relatively small, and a state

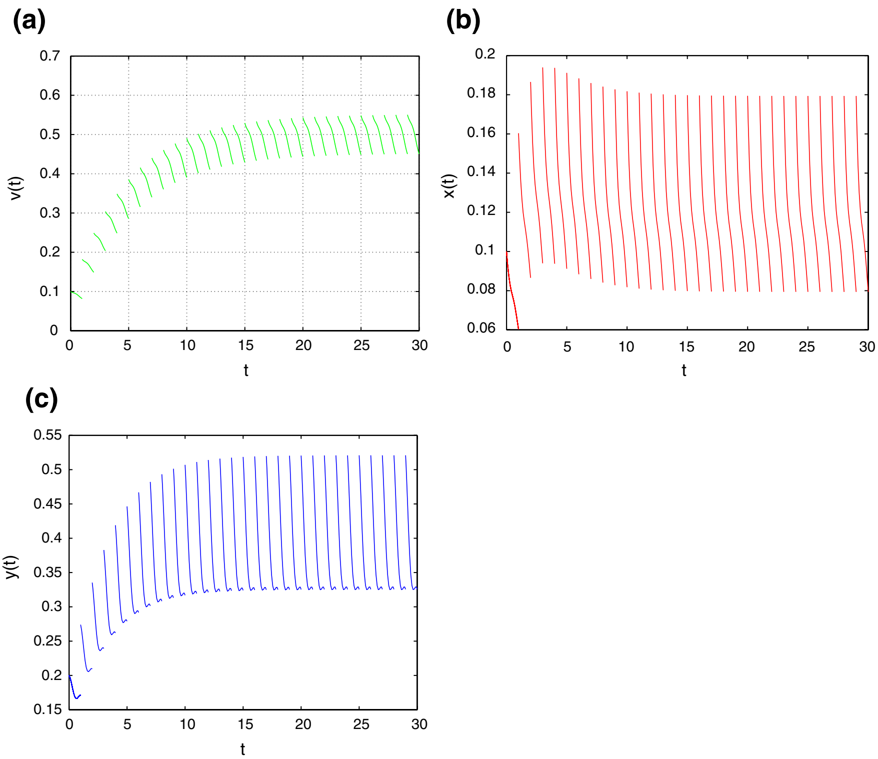

**Fig. 3** Asymptotic behaviors of systems (4.1) and (3.2) with $K_1(t) = -0.2 + 0.1 \sin 2\pi t$, $K_2(t) = 0.15 + 0.1 \cos 2\pi t$, $p \equiv 0.1$, $b_k \equiv 0.6$. **a** Time-series of the species $v$. **b** Time-series of the species $x$. **c** Time-series of the species $y$

of coexistence occurs, which is in line with reality from a biological point of view. When $K_1(t) = -0.2 + 0.1 \sin 2\pi t$ (that is, the species $x$ lives in a weak environment) and $p \equiv 0.1$, system (3.2) has a unique positive globally asymptotically stable 1-periodic solution denoted by $\bar{v}_1(t)$ (see Fig. 3a, here $\bar{v}_1(t) < 0.55$). Furthermore, when $K_2(t) = 0.15 + 0.1 \cos 2\pi t$ and $b_k \equiv 0.6$, it is easy to verify that the assumptions of (1) and (2) in Theorem 3.1 ($\lambda_1 > 1$) are satisfied, so both $x$ and $y$ are permanent (see Fig. 3b, c). The result shows that the impulsive control strategy can save the species $x$ from extinction although the species lives in a weak environment. When $K_1(t) = 0.08 + 0.1 \sin 2\pi t$, $K_2(t) = 0.15 + 0.1 \cos 2\pi t$, $p \equiv 0.5$, and $b_k \equiv -0.2$, we can verify that the conditions of Theorem 3.2 are satisfied. Then the species $x$ will be driven to stabilize at a positive periodic solution of an impulsive logistic system while the species $y$ tend toward extinction in the end. (see Fig. 4a–c). This result shows that the human activities result in the extinction of the species $y$, and hence it is very important to plan a suitable harvesting or stocking policy in order to make both species coexist.

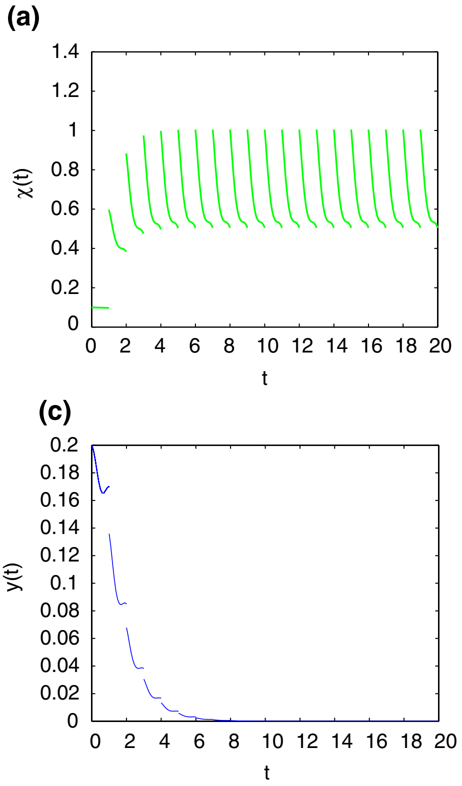

**Fig. 4** Asymptotic behaviors of systems (4.1) and (3.30) with $K_1(t) = 0.08 + 0.1 \sin 2\pi t$, $K_2(t) = 0.15 + 0.1 \cos 2\pi t$, $p \equiv 0.5$, $b_k \equiv -0.2$. **a** Time-series of the species $\chi$. **b** Time-series of the species $x$ and $\chi$. **c** Time-series of the species $y$

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## References

1. T.G. Hallam, C.E. Clark, Nonautonomous logistic equations as models of populations in a deteriorating environment. J. Theor. Biol. **93**, 303–311 (1982)
2. H.I. Freedman, J.B. Schukla, Models for the effect of toxicant in single species and predator–prey systems. J. Math. Biol. **30**, 15–30 (1990)
3. J.M. Smith, *Models in Ecology* (Cambridge University, Cambridge, 1974)
4. J. Chattopadhyay, Effect of toxic substances on a two-species competitive system. Ecol. Model. **84**, 287–289 (1996)
5. A. Mukhopadhyay, J. Chattopadhyay, P.K. Tapaswi, A delay differential model of plankton allelopathy. Math. Biosci. **149**, 167–189 (1998)

6. B. Liu, F.M. Tao, The optimal pulse harvesting policy on a single-species population model with birth pulses in a polluted environment. Adv. Complex Syst. **10**, 173–196 (2007)
7. B. Liu, L.S. Chen, Y.J. Zhang, The effects of impulsive toxicant input on a population in a polluted environment. J. Bio. Syst. **11**, 265–274 (2003)
8. A. Bernatik, W. Zimmerman, M. Pitt, M. Strizik, V. Nevrly, Z. Zelinger, Modelling accidental releases of dangerous gases into the lower troposphere from mobile sources. Process Safety Environ. Protect. **86**, 198–207 (2008)
9. P. Kassomenos, A. Karayannis, I. Panagopoulos, S. Karakitsios, M. Petrakis, Modelling the dispersion of a toxic substance at a workplace. Environ. Model. Software **23**, 82–89 (2008)
10. Ch. Papadimitriou, G. Palaska, M. Lazaridou, P. Samaras, G.P. Sakellaropoulos, The effects of toxic substances on the activated sludge microfauna. Desalination **211**, 177–191 (2007)
11. R.R. Sarkar, B. Mukhopadhyay, R. Bhattacharyya, S. Banerjee, Time lags can control algal bloom in two harmful phytoplankton–zooplankton system. Appl. Math. Comput. **186**, 445–459 (2007)
12. Z. Li, F.D. Chen, Extinction in two dimensional nonautonomous Lotka-Volterra systems with the effect of toxic substances. Appl. Math. Comput. **182**, 684–690 (2006)
13. X.Y. Song, L.S. Chen, Periodic solution of a delay differential equation of plankton allelopathy. Acta. Math. Sci. Ser. A **23**, 8–13 (2003) (Chinese)
14. Z.J. Liu, L.S. Chen, Periodic solution of a two-species competitive system with toxicant and birth pulse. Chaos Solitons Fract. **32**, 1703–1712 (2007)
15. C.W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources* (Wiley, New York, 1976)
16. B.S. Goh, *Management and Analysis of Biological Populations* (Elsevier Scientific, Netherlands, 1980)
17. D.D. Bainov, P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications* (Longman Scientific and Technical, Burnt Mill, 1993)
18. V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations* (World Scientific, Singapore, 1989)
19. A. D'onofrio, Stability properties of pulse vaccination strategy in SEIR epidemic model. Math. Biosci. **179**, 57–72 (2002)
20. X.N. Liu, L.S. Chen, Global dynamics of the periodic logistic system with periodic impulsive perturbations. J. Math. Anal. Appl. **289**, 279–291 (2004)
21. W.B. Wang, J.H. Shen, Z.G. Luo, Partial survival and extinction in two competing species with impulses. Nonlinear Anal. RWA. doi:10.1016/j.nonrwa.2007.11.012
22. B. Liu, L.S. Chen, The periodic competing Lotka-Volterra model with impulsive effect. Math. Med. Biol. **21**, 129–145 (2004)
23. A. Lakmeche, O. Arino, Bifurcation of non trivial periodic solutions of impulsive differential equations arising chemotherapeutic treat. Dynam. Contin. Discrete Impuls. Syst. **7**, 165–187 (2000)
24. F.Y. Wang, G.P. Pang, J. Hui, Analysis of a Monod-Haldene type food chain chemostat with seasonally variably pulsed input and washout. J. Math. Chem. **43**, 601–619 (2008)
25. S.L. Sun, L.S. Chen, Dynamic behaviors of Monod type chemostat model with impulsive perturbation on the nutrient concentration. J. Math. Chem. **42**, 837–847 (2007)
26. L.S. Sun, L.S. Chen, Complex dynamics of a chemostat with variable yield and periodically impulsive perturbation on the substrate. J. Math. Chem. **43**, 338–349 (2008)
27. J. Hui, L.S. Chen, Dynamic complexities in ratio-dependent predator–prey ecosystem models with birth pulse and pesticide pulse. Int. J. Bifur. Chaos Appl. Sci. Eng. **14**, 2893–2903 (2004)
28. S.Y. Tang, L.S. Chen, Density-dependent birth rate, birth pulses and their population dynamic consequences. J. Math. Biol. **44**, 185–199 (2002)
29. S.Y. Tang, L.S. Chen, Multiple attractors in stage-structured population models with birth pulses. Bull. Math. Biol. **65**, 479–495 (2003)
30. W.B. Wang, J.H. Shen, J.J. Nieto, Permanence and periodic solution of predator–prey system with Holling type functional response and impulses. Discrete Dyn. Nat. Soc. Art. ID 81756, 15 pp (2007)
31. S.Y. Tang, L.S. Chen, The periodic predator–prey Lotka-Volterra model with impulsive effect. J. Mech. Med. Biol. **3&4**, 267–296 (2002)